Average (Mean) values calculation for Random Walk Problem :-

From the ~~ka~~ earlier lecture notes, we have the probability of taking n_1 steps to the right & of n_2 steps left where $n_2 = N - n_1$

$$W(n_1) = \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} \quad \text{—— (1)}$$

Let us calculate $\sum_{n_1=0}^{N} W(n_1)$

$$\sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} = (p+q)^N$$

Where we have used the ~~bto~~ binomial theorem

or $\sum_{n_1=0}^{N} W(n_1) = (p+q)^N = 1^N = 1$

$$\boxed{\sum_{n_1=0}^{N} W(n_1) = 1} \quad \text{—— (2)}$$

$\Rightarrow$ Probability of making any number of right steps between 0 and N is unity $\Rightarrow$ $W(n_1)$ is normalized

Next, Mean of n_1 is given by

$$\bar{n}_1 = \sum_{n_1=0}^{N} W(n_1) n_1 = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} \times n_1 \quad \text{—— (3)}$$

Let us use some mathematical trick to simplify the above expression.

We can write $n_1 p^{n_1} = p\frac{\partial}{\partial p}(p^{n_1})$

using above expression in (3)

$$\bar{n}_1 = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!}\left[p\frac{\partial}{\partial p}(p^{n_1})\right] q^{N-n_1}$$

or $$\bar{n}_1 = p\frac{\partial}{\partial p}\left[\underbrace{\sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}}_{\text{This is binomial expansion of } (p+q)^N}\right]$$

$$\bar{n}_1 = p\frac{\partial}{\partial p}(p+q)^N = pN(p+q)^{N-1}$$

$$\boxed{\bar{n}_1 = pN(p+q)^{N-1}} \longrightarrow \text{True for arbitrary values of } p \,\&\, q$$

for specific case, ~~$q = 1-p$~~ $p+q=1$, Then

$$\boxed{\bar{n}_1 = Np} \quad \text{—— (4)}$$

H.W. Calculate $\bar{n}_2$ using the above procedure

Here we can guess $\bar{n}_2$ from eq^n (4). p is the probability of making a right step, the mean number of right step is given by total no. of steps multiplied by the probability p. Thus.

$$\boxed{\bar{n}_2 = Nq} \quad \text{—— (5)}$$

and $\bar{n}_1 + \bar{n}_2 = Np + Nq = N(p+q) = N$

Displacement $m = n_1 - n_2$

Mean displacement $\Rightarrow \bar{m} = \overline{n_1 - n_2} = \bar{n}_1 - \bar{n}_2$

or $$\boxed{\bar{m} = N(p-q)} \quad \text{—— (6)}$$

If, for special case, $p = q$

$$\bar{m} = 0$$

Dispersion:

$$\overline{(\Delta n_1)^2} = \overline{(n_1 - \bar{n}_1)^2}$$
$$= \overline{n_1^2 - 2n_1\bar{n}_1 + \bar{n}_1^2}$$
$$= \overline{n_1^2} - 2\bar{n}_1^2 + \bar{n}_1^2$$
$$\overline{(\Delta n_1)^2} = \overline{n_1^2} - \bar{n}_1^2$$

From (1) $$\overline{n_1^2} = \sum_{n_1=0}^{N} W(n_1) n_1^2$$

$$= \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} \times n_1^2$$

Again $$n_1^2 p^{n_1} = n_1 \left(p\frac{\partial}{\partial p}\right)(p^{n_1}) = \left(p\frac{\partial}{\partial p}\right)^2 (p^{n_1})$$

or $$\overline{n_1^2} = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} \left(p\frac{\partial}{\partial p}\right)^2 (p^{n_1}) q^{N-n_1}$$

$$= \left(p\frac{\partial}{\partial p}\right)^2 \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}$$

$$= \left(p\frac{\partial}{\partial p}\right)^2 (p+q)^N$$

$$= p\frac{\partial}{\partial p}\left[pN(p+q)^{N-1}\right]$$

$$\boxed{\overline{n_1^2} = p\left[N(p+q)^{N-1} + pN(N-1)(p+q)^{N-2}\right]} \quad \text{— (7)}$$

For specific case $p+q=1$

$$\overline{n_1^2} = p[N + pN(N-1)] = Np[1 + pN - p]$$

$$= Np[Np + q] = (Np)^2 + Npq$$

$$\boxed{\overline{n_1^2} = \bar{n}_1^2 + Npq} \quad \text{— (8)} \qquad \{\text{since } \bar{n}_1 = Np$$

Now $$\overline{(\Delta n_1)^2} = \overline{n_1^2} - \bar{n}_1^2 = \cancel{\bar{n}_1^2} + Npq - \cancel{\bar{n}_1^2}$$

$$\boxed{\overline{(\Delta n_1)^2} = Npq}$$

Relative width distribution $R_W = \dfrac{\sqrt{\overline{(\Delta n_1)^2}}}{\overline{n_1}}$

$$R_W = \frac{\sqrt{Npq}}{Np} = \frac{1}{\sqrt{N}}\sqrt{\frac{p}{q}}$$

$$\boxed{R_W = \frac{1}{\sqrt{N}}\sqrt{\frac{p}{q}}}$$

For special case $p = q = \frac{1}{2}$

$$\boxed{R_W = \frac{1}{\sqrt{N}}}$$

Some key points $\overline{n_1} \rightarrow$ increases with N as

$$\overline{n_1} \propto N$$

and R_W decreases with N as

$$\boxed{R_W \propto \frac{1}{\sqrt{N}}}$$

H.W. Calculate dispersion for displacement m

Ans is $\boxed{\overline{(\Delta m)^2} = 4Npq}$ prove it